Next-nearest-neighbor spin-spin and chiral-spin correlation functions in a generalized *XXX* chain

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We develop a simple technique for calculation of next-to-nearest-neighbor spin-spin and chiral-spin correlation functions in an inhomogeneous *XXX* model. Exact expression of the chiral-spin order parameter as a function of the model parameter, ω, is analytically found. Using the same method we also calculate the next-to-nearest-neighbor spin-spin correlation function. In the limit $\omega \to 0$ it reproduces the known result for the vacuum expectation value of the next-to-nearest-neighbor spins in the standard Heisenberg spin chain. The technique is simple and can be extended for calculation of next-to-next-to-nearest-neighbor correlation functions as well as for calculation of correlation functions in the *XXZ* model.

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I. INTRODUCTION

Presently, calculation of correlation functions in strongly correlated electron systems is one of the most important tasks in physics of low dimensions. Correlation functions of spins at *large distances* are directly related to the observable quantities. They define a set of critical indices identifying the universality classes of different phases.[1–4] Notably, correlation functions of spins in the homogeneous Heisenberg spin chain (*XXX* model) at short distances are not directly related to the universality class of the phase, but, as it appeared,[5] they play an important role in the perturbative investigations of ladder models that are not integrable. An important example is the Haldane ladder model[6] that can be approached as a Heisenberg spin chain perturbed with the chiral-spin order operator, $\vec{\sigma}_n(\vec{\sigma}_{n+1}\times\vec{\sigma}_{n+2})$ (which was considered earlier in connection with spin-liquid ordered phase in Ref. 7), and by the product of next-to-nearest-neighbor spins,[5] $\langle\vec{\sigma}_n\vec{\sigma}_{n+2}\rangle$. In order to investigate the free energy of this long term attractive model and analyze the phase space, one needs to know the correlation function of the next-to-nearest-neighbor spins and the chiral-spin order parameter $\chi_n=\langle\vec{\sigma}_n(\vec{\sigma}_{n+1}\times\vec{\sigma}_{n+2})\rangle$.

The aim of the present work is the calculation of the above-mentioned correlation functions for a generalized *XXX* model[8–13] by a simple technique. This model is defined and studied in Sec. II. The studies of Sec. II involve analysis of the Hamiltonian(s) and other conserved currents, construction of the transfer matrix, and obtaining the set of Bethe equations that describe the energy spectrum. Two particular cases of the generalized *XXX* model are the ladder systems with N sites at each chain given by Hamiltonian operators

$$\mathcal{H} = 2\sum_{n=1}^{2N}[\vec{\sigma}_n\vec{\sigma}_{n+1}-1] + \omega^2\sum_{n=1}^{2N}[\vec{\sigma}_n\vec{\sigma}_{n+2}-1] + \omega\sum_{n=1}^{2N}(-1)^n\vec{\sigma}_{n-1}(\vec{\sigma}_n\times\vec{\sigma}_{n+1}) \tag{1}$$

and

$$\mathcal{H}' = \omega^2\sum_{n=1}^{2N}(-1)^n[\vec{\sigma}_n\vec{\sigma}_{n+2}-1] - \omega\sum_{n=1}^{2N}\vec{\sigma}_{n-1}(\vec{\sigma}_n\times\vec{\sigma}_{n+1}). \tag{2}$$

As a manifestation of the effective workability of the developed technique in Sec. III we first calculate the exact correlation functions of the next-to-nearest-neighbor spins, $\xi(\omega)=\langle\vec{\sigma}_n\vec{\sigma}_{n+2}-1\rangle$, as a function of ω, which for the Hamiltonian Eq. (1) has the following asymptotic behavior:

$$\xi(\omega) = \begin{cases} -16\log(2)+9\zeta(3)-1.41\omega^2 & \text{for } \omega \ll 1, \\ -2.77+\dfrac{1.77}{\omega^2} & \text{for } \omega \gg 1. \end{cases} \tag{3}$$

Here $\zeta(x)$ is the Riemann zeta function. In the limit $\omega=0$, we reproduce the known result for the expectation value of the next-to-nearest-neighbor exchange operator in the standard *XXX* chain, first obtained by involving the Hubbard model.[14]

More recently, within the general approach of multiple integral representation of correlation functions that was formulated in Ref. 15 and investigated further in Refs. 16 and 17, correlation functions $\langle\vec{\sigma}_n\vec{\sigma}_{n+3}\rangle$ and $\langle\vec{\sigma}_n\vec{\sigma}_{n+4}\rangle$ were evaluated.[18,19] These functions were evaluated for the *XXX* model in zero magnetic field. One of the advantages of our technique is the possibility to include also external magnetic field. This can be done by involving the magnetic field into the integral equations for the densities. In some cases such integral equations can be solved by approximate methods.[20]

Our main result, however, is the analytical calculation of the chiral-spin order parameter, $\langle\vec{\sigma}_n(\vec{\sigma}_{n+1}\times\vec{\sigma}_{n+2})\rangle=(-1)^n\chi(\omega)$, as a function of the staggering parameter, ω. The full expression and the derivation of χ_n are presented in Sec. III. Here we present only the asymptotes of the function $\chi(\omega)$,

$$\chi(\omega) = \begin{cases} [8\gamma+8\psi(1/2)-\psi''(1/2)+\psi''(1)]\omega & \text{for } \omega \ll 1, \\ \dfrac{3.545}{\omega} & \text{for } \omega \gg 1, \end{cases} \tag{4}$$

where γ is the Euler constant, the function ψ is the Digamma function, and the coefficient of linearity for small ω is ≈ 3.33.

We would like to note that this correlation function was previously investigated numerically[12] and the match between curves plotted from our exact result (Fig. 2) and from the numerical simulations is perfect.

II. INHOMOGENEOUS CHAIN AND A FAMILY OF COMMUTING OPERATORS

In one dimension, there is a variety of quantum exactly solvable models of interacting spins. Some of these models involve interactions between nearest-neighbor spins and also spins that are far from each other. Usually the further neighbor interactions and other non-localities come from additional anisotropy parameters. We will consider a family of models with nearest-neighbor, next-to-nearest neighbor, and triangular (zigzag) interactions, which stem from the transfer matrix with the shift of the spectral parameters at each second sites:

$$\Theta(\lambda;\omega)=L_{2N,a}(\lambda)L_{2N-1,a}(\lambda-\omega)\cdots L_{2,a}(\lambda)L_{1,a}(\lambda-\omega). \tag{5}$$

Here $L_{a,b}(\lambda)$ obeys the rational Yang-Baxter relations

$$R_{a_1,a_2}(\lambda-\mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu)=L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu) \tag{6}$$

with rational

$$L_{n,a}(\lambda)=(\lambda-i/2)\mathbb{1}_{\mathrm{n,a}}+iP_{\mathrm{n,a}},$$

$$R_{a,b}(\lambda)=\lambda\mathbb{1}_{\mathrm{a,b}}+iP_{\mathrm{a,b}}. \tag{7}$$

The permutation operator is given in terms of Pauli matrices as

$$P_{a,b}=\frac{1}{2}\left(\mathbb{1}_{\mathrm{a}}\otimes\mathbb{1}_{\mathrm{b}}+\sum_{\alpha}\sigma^{\alpha}\otimes\sigma^{\alpha}\right).$$

With this construction, one has a commuting one parametrical family of transfer matrices $\tau(\lambda;\omega)=tr_a\Theta(\lambda;\omega)$:

$$[\tau(\lambda;\omega),\tau(\mu;\omega)]=0.$$

This is a well-known picture, see, e.g., Refs. 8–10, while we want to apply this in a slightly different content. In order to be shorter we will miss some proofs and refer the reader to the nice review by Faddeev.[8]

The interesting feature of the transfer matrix $\Theta(\lambda;\omega)$, given by Eq. (5), is that instead of the usual *XXX* Hamiltonian this yields two different local Hamiltonian operators, H_1 and H_2, that are proportional to the logarithmic derivative of τ at two different points: $\lambda=i/2$ and $\lambda=\omega+i/2$. Respectively, their explicit forms are

$$\begin{aligned}H_1=&\ 2i(1+\omega^2)\partial_\lambda\ln\tau|_{\lambda=i/2}-N(2+\omega^2-2i\omega)\\&=\sum_{n=1}^{2N}[\vec{\sigma}_n\vec{\sigma}_{n+1}-1]\\&+\sum_{k=1}^{N}\{\omega^2[\vec{\sigma}_{2k}\vec{\sigma}_{2k+2}-1]-\omega\vec{\sigma}_{2k}(\vec{\sigma}_{2k+1}\times\vec{\sigma}_{2k+2})\},\end{aligned} \tag{8}$$

$$\begin{aligned}H_2=&\ 2i(1+\omega^2)\partial_\lambda\ln\tau|_{\lambda=i/2+\omega}-N(2+\omega^2+2i\omega)\\&=\sum_{n=1}^{2N}[\vec{\sigma}_n\vec{\sigma}_{n+1}-1]+\sum_{k=1}^{N}\{\omega^2[\vec{\sigma}_{2k-1}\vec{\sigma}_{2k+1}-1]\\&+\omega\vec{\sigma}_{2k-1}(\vec{\sigma}_{2k}\times\vec{\sigma}_{2k+1})\}.\end{aligned} \tag{9}$$

These operators are commuting as they belong to the same commuting family. It is straightforward that the Hamiltonian operators $\mathcal{H}=H_1-H_2$ and $\mathcal{H}'=H_1+H_2$ are exactly diagonalizable in the same framework. Their explicit forms are given by Eqs. (1) and (2).

Let *quasishift operators* be the monodromy matrices at the points $\lambda=i/2$ and $\lambda=i/2+\omega$:

$$U_+=tr_a\Theta(i/2;\omega),$$

$$U_-=tr_a\Theta(i/2+\omega;\omega). \tag{10}$$

They obey the relation

$$(1+\omega^2)^N V^2=U_+U_-=e^{iP}, \tag{11}$$

where

$$V=P_{1,2}P_{2,3}\cdots P_{2N-1,2N} \tag{12}$$

is a shift on one unit $n\to n+1$, and P is the physical momentum, which governs the shift $n\to n+2$. Being defined in this way, the quasishift operators commute with the whole family of transfer matrices $\tau(\mu;\omega)$.

Derivation of the Bethe ansatz equations (BAE) for $\tau(\mu;\omega)$ is standard, see e.g., Ref. 8. Starting from the reference state with all $2N$ spins up one can generate the eigenvectors of the transfer matrix in the sector with M overturned spins, parametrized by M complex rapidities λ_n which obey BAE

$$\left(\frac{(\lambda_n+i/2)(\lambda_n-\omega+i/2)}{(\lambda_n-i/2)(\lambda_n-\omega-i/2)}\right)^N=\prod_{k\neq n}^{M}\frac{\lambda_n-\lambda_k+i}{\lambda_n-\lambda_k-i}. \tag{13}$$

The eigenvalues of the transfer matrix $\tau(\mu;\omega)$ have the form

$$\begin{aligned}t(\mu)=&\left[\left(\mu+\frac{i}{2}\right)\left(\mu-\omega+\frac{i}{2}\right)\right]^N\prod_{n=1}^{M}\frac{\mu-\lambda_n-i}{\mu-\lambda_n}\\&+\left[\left(\mu-\frac{i}{2}\right)\left(\mu-\omega-\frac{i}{2}\right)\right]^N\prod_{n=1}^{M}\frac{\mu-\lambda_n+i}{\mu-\lambda_n}.\end{aligned} \tag{14}$$

This gives the quasiparticle momentum in the form

$$e^{iP}=\prod_{n=1}^{M}\frac{(\lambda_n+i/2)(\lambda_n-\omega+i/2)}{(\lambda_n-i/2)(\lambda_n-\omega-i/2)}, \tag{15}$$

and eigenvalues of H_1 and H_2 as follows:

$$E_1(\{\lambda\},\omega)=-2(1+\omega^2)\sum_n\frac{1}{\lambda_n^2+1/4},$$

$$E_2(\{\lambda\},\omega) = -2(1+\omega^2)\sum_n \frac{1}{(\lambda_n-\omega)^2+1/4}. \quad (16)$$

The corresponding eigenenergies of Eqs. (1) and (2) are E_1+E_2 and E_1-E_2, respectively. Now the picture is in some sense complete and one is in position to infer the thermodynamics of these models, based on the BAE and the energy relations. The particular Hamiltonian Eq. (1) was introduced in Ref. 11 and analyzed in detail.[10,12] It has a singlet ground state with massless excitations. By involving a magnetic field with the Zeeman coupling, the system undergoes two phase transitions; two critical phases with different universality classes are discussed in Ref. 21. The *XXZ* generalization of the model (2) was defined and investigated in Refs. 13 and 22.

III. CHIRAL-SPIN AND OTHER CORRELATION FUNCTIONS

It is well-known that the expectation values of operators not commuting with the Hamiltonian are not easily accessible within the framework of Bethe ansatz. In the case of the inhomogeneous chain under consideration we have the additional parameter ω, which breaks the translational invariance $n\to n+1$ and gives a possibility to calculate some simplest expectation values that are valid also in the limit $\omega\to 0$, corresponding to the well-known *XXX* case. For our purposes we need the eigenvalues of quasitranslation operators, which follow from Eq. (14) and the definition (10):

$$u_+ = (-1)^N(1+i\omega)^N\prod_n \frac{\lambda_n+i/2}{\lambda_n-i/2},$$

$$u_- = (-1)^N(1-i\omega)^N\prod_n \frac{\lambda_n-\omega+i/2}{\lambda_n-\omega-i/2}. \quad (17)$$

One can differentiate Eq. (10) with respect to ω and get the relations

$$\partial_\omega U_+ = \sum_{k=1}^{N} \frac{iP_{2k,2k-1}+\omega}{1+\omega^2}U_+ = U_+\sum_{k=1}^{N} \frac{iP_{2k,2k+1}+\omega}{1+\omega^2}, \quad (18)$$

$$\partial_\omega U_- = -\sum_{k=1}^{N} \frac{iP_{2k,2k+1}-\omega}{1+\omega^2}U_- = -U_-\sum_{k=1}^{N} \frac{iP_{2k,2k-1}-\omega}{1+\omega^2}. \quad (19)$$

Let us use the relation that is always valid when one has a parameter-dependent operator $\hat{O}(\eta)$ with the spectrum $o_n(\eta)$ and normalized eigenstates $|n\rangle$,

$$\langle n|\partial_\eta \hat{O}(\eta)|n\rangle = \partial_\eta o_n(\eta). \quad (20)$$

Then we find that

$$\langle\{\mu\}|\vec{\sigma}_n\vec{\sigma}_{n+1}|\{\mu\}\rangle = \mp\frac{2}{N}i(1+\omega^2)\partial_\omega \ln[u_\pm(\omega)] + 2i\omega - 1$$
$$= 1 - 2\frac{(1+\omega^2)}{N}\sum_{m=1}^{M}\frac{\partial_\omega\mu_m}{\mu_m^2+1/4}, \quad (21)$$

where $\{\mu\}$ is any set of BAE solution. Given in the above form, it is readily calculable. It does not depend on n, even on the parity of n and shows that the model Eq. (1) can not have a dimerized phase.

The next expectation value that we are going to evaluate is the next-to-nearest neighbor (NNN) exchange, $\langle\{\mu\}|\vec{\sigma}_n\vec{\sigma}_{n+2}|\{\mu\}\rangle$. For this purpose we divide the two Hamiltonian operators Eq. (8) by ω and apply Eq. (20). Subtracting contributions of nearest-neighbor terms with the use of Eq. (21) we obtain for even or odd site numbers the following relations:

$$\langle\{\mu\}|\vec{\sigma}_{2k}\vec{\sigma}_{2k+2}-1|\{\mu\}\rangle$$
$$= \frac{1}{N}\partial_\omega\frac{E_1(\{\mu\},\omega)}{\omega} + \frac{2}{\omega^2}\langle\{\mu\}|\vec{\sigma}_n\vec{\sigma}_{n+1}-1|\{\mu\}\rangle$$
$$= -\frac{4}{N}\sum_{m=1}^{M}\left\{\frac{1+\omega^2}{\omega^2}\frac{\partial_\omega\mu_m}{\mu_m^2+\frac{1}{4}} + \frac{\omega^2-1}{2\omega^2}\frac{1}{\mu_m^2+\frac{1}{4}} - \frac{1+\omega^2}{\omega}\frac{\mu_m\partial_\omega\mu_m}{\left(\mu_m^2+\frac{1}{4}\right)^2}\right\} \quad (22)$$

and

$$\langle\{\mu\}|\vec{\sigma}_{2k-1}\vec{\sigma}_{2k+1}-1|\{\mu\}\rangle$$
$$= \frac{1}{N}\partial_\omega\frac{E_2(\{\mu\},\omega)}{\omega} + \frac{2}{\omega^2}\langle\{\mu\}|\vec{\sigma}_n\vec{\sigma}_{n+1}-1|\{\mu\}\rangle$$
$$= -\frac{4}{N}\sum_{m=1}^{M}\left\{\frac{1+\omega^2}{\omega^2}\frac{\partial_\omega\mu_m}{\mu_m^2+\frac{1}{4}} + \frac{\omega^2-1}{2\omega^2}\frac{1}{(\mu_m-\omega)^2+\frac{1}{4}} - \frac{1+\omega^2}{\omega}\frac{(\mu_m-\omega)(\partial_\omega\mu_m-1)}{\left[(\mu_m-\omega)^2+\frac{1}{4}\right]^2}\right\}. \quad (23)$$

It is yet unknown how to perform such summations in the case of finite N and M analytically. Instead we can evaluate the sums in the important case of the thermodynamic limit,

$$N\to\infty, \quad M\to\infty, \quad \frac{M}{N} = \text{const}, \quad (24)$$

when solutions of Eq. (13) form bound states called strings.[14] In our rational case strings with arbitrary length n are possible. Consider the case where one has M_n bound states of n strings,

$$\lambda_\alpha^{n,j} = \lambda_\alpha^n + \frac{i}{2}(n+1-2j) + O[\exp(-|\delta|N)],$$

$$\alpha = 1, \dots, M_n, \quad j = 1, \dots, n, \tag{25}$$

with real parameters of centers λ_α^n. Then one can take the product of n BAE for the same n string and obtain an equation for real centers. The logarithm of these equations gives

$$\theta(\lambda_\alpha^n/n) + \theta[(\lambda_\alpha^n - \omega)/n] = \frac{2\pi}{N} I_\alpha^n + \frac{1}{N} \sum_{(m,\beta)\neq(n,\alpha)} \Theta_{nm}(\lambda_\alpha^n - \lambda_\beta^m), \tag{26}$$

where $\theta(\lambda) \equiv 2\tan^{-1}[2\lambda]$ and

$$\Theta_{nm}(\lambda) \equiv \begin{cases} \theta\left(\frac{\lambda}{|n-m|}\right) + 2\theta\left(\frac{\lambda}{|n-m|+2}\right) + \cdots + 2\theta\left(\frac{\lambda}{n+m-2}\right) + \theta\left(\frac{\lambda}{n+m}\right) & \text{for } n \neq m, \\ 2\theta\left(\frac{\lambda}{2}\right) + 2\theta\left(\frac{\lambda}{4}\right) + \cdots + 2\theta\left(\frac{\lambda}{2n-2}\right) + \theta\left(\frac{\lambda}{2n}\right) & \text{for } n = m. \end{cases} \tag{27}$$

I_α^n is an integer (half-odd integer) if $N-M_n$ is odd (even) and satisfies

$$|I_\alpha^n| \leq \frac{1}{2}\left(N - 1 - \sum_{m=1}^{\infty} t_{nm} M_m\right),$$

$$t_{nm} \equiv 2Min(n,m) - \delta_{nm}. \tag{28}$$

In the thermodynamic limit (24), it is convenient to define distribution functions of n strings $\rho_n(\lambda)$ and holes of n string $\tilde{\rho}_n(\lambda)$; the number of strings and holes between λ and $\lambda+d\lambda$ is $\rho_n(\lambda)Nd\lambda$ and $\tilde{\rho}_n(\lambda)Nd\lambda$, respectively. From Eqs. (26) one obtains a system of integral equations

$$a_n(\lambda) + a_n(\lambda - \omega) = \rho_n(\lambda) + \tilde{\rho}_n(\lambda) + \sum_m \int_{-\infty}^{\infty} T_{nm}(\lambda - \mu)\rho_m(\mu)d\mu, \tag{29}$$

where $T_{nm}(\lambda)$ is a function defined by

$$T_{nm}(\lambda) \equiv \begin{cases} a_{|n-m|}(\lambda) + 2a_{|n-m|+2}(\lambda) + 2a_{|n-m|+4}(\lambda) + \cdots + 2a_{n+m-2}(\lambda) + a_{n+m}(\lambda) & \text{for } n \neq m, \\ 2a_2(\lambda) + 2a_4(\lambda) + \cdots + 2a_{2n-2}(\lambda) + a_{2n}(\lambda) & \text{for } n = m, \end{cases} \tag{30}$$

and $a_n(\lambda)$ is a function defined by

$$a_n(\lambda) \equiv \frac{1}{\pi} \frac{2n}{4\lambda^2 + n^2}.$$

In order to describe ω derivatives of λ_α^n in the thermodynamic limit, we introduce a function, F_n, as

$$\lim_{N\to\infty} \partial_\omega \lambda_\alpha^n F_n(\lambda, \omega). \tag{31}$$

For briefness we will miss the explicit ω dependence of F. In order to find a characteristic integral equation for this function, one can differentiate Eq. (26) with respect to ω and use Eq. (29). In this way one finds:

$$a_n(\lambda - \omega) = F_n(\lambda)[\rho_n(\lambda) + \tilde{\rho}_n(\lambda)] + \sum_m \int_{-\infty}^{\infty} T_{nm}(\lambda - \mu)F_m(\mu)\rho_m(\mu)d\mu. \tag{32}$$

Now we can rewrite the right-hand side sums in Eq. (22) in terms of integrals:

$$\langle\{\mu\}|\vec{\sigma}_{2k}\vec{\sigma}_{2k+2} - 1|\{\mu\}\rangle = -8\pi \sum_{n=1}^{\infty} \int \rho_n(\mu)d\mu \left\{ \frac{1+\omega^2}{\omega^2} a_n(\mu)F_n(\mu) + \frac{\omega^2 - 1}{2\omega^2} a_n(\mu) + \frac{1+\omega^2}{2\omega} a_n'(\mu)F_n(\mu) \right\} \tag{33}$$

and

$$\langle\{\mu\}|\vec{\sigma}_{2k-1}\vec{\sigma}_{2k+1}-1|\{\mu\}\rangle=-8\pi\sum_{n=1}^{\infty}\int\rho_n(\mu)d\mu\left\{\frac{1+\omega^2}{\omega^2}a_n(\mu)F_n(\mu)+\frac{\omega^2-1}{2\omega^2}a_n(\mu-\omega)+\frac{1+\omega^2}{2\omega}a_n'(\mu-\omega)[F_n(\mu)-1]\right\}. \tag{34}$$

Evaluation of the expectation value of triple interaction terms, the chiral-spin order parameter, can be done in a similar way. The answer is

$$\langle\{\mu\}|\vec{\sigma}_{2k}(\vec{\sigma}_{2k+1}\times\vec{\sigma}_{2k+2})|\{\mu\}\rangle=-4\pi\sum_{n=1}^{\infty}\int\rho_n(\mu)d\mu\left\{4\frac{1+\omega^2}{\omega}a_n(\mu)F_n(\mu)-\frac{2}{\omega}a_n(\mu)+(1+\omega^2)a_n'(\mu)F_n(\mu)\right\} \tag{35}$$

and

$$\langle\{\mu\}|\vec{\sigma}_{2k-1}(\vec{\sigma}_{2k}\times\vec{\sigma}_{2k+1})|\{\mu\}\rangle=4\pi\sum_{n=1}^{\infty}\int\rho_n(\mu)d\mu\left\{4\frac{1+\omega^2}{\omega}a_n(\mu)F_n(\mu)-\frac{2}{\omega}a_n(\mu-\omega)+(1+\omega^2)a_n'(\mu-\omega)[F_n(\mu)-1]\right\}. \tag{36}$$

For definiteness, let us evaluate the NNN expectation value for the ground state of the Hamiltonian Eq. (1). For this state, the densities are found to be zero for all the n strings with $n=2,3,\ldots$ and 1-holes.[12] The system Eq. (29) reduces to the following simple integral equation for $n=1$:

$$a_1(\lambda)+a_1(\lambda-\omega)=\rho_1(\lambda)+\int_{-\infty}^{\infty}T_{11}(\lambda-\mu)\rho_1(\mu)d\mu. \tag{37}$$

Its solution

$$\rho_1(\lambda)=\frac{1}{2\cosh\pi\lambda}+\frac{1}{2\cosh\pi(\lambda-\omega)}, \tag{38}$$

and the corresponding solution to Eq. (32)

$$F_1(\lambda)\rho_1(\lambda)=\frac{1}{2\cosh\pi(\lambda-\omega)} \tag{39}$$

can be found by the Fourier transform. The integrals in Eq. (33) can be easily transformed to the following expression for the NNN expectation value:

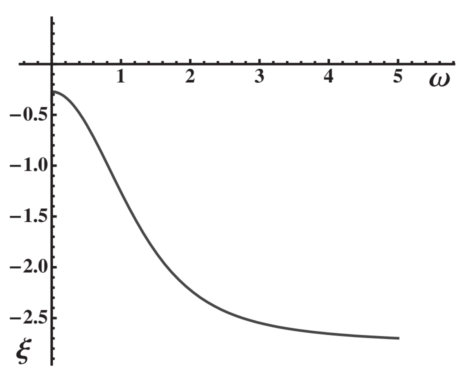

FIG. 1. Next-to-nearest neighbor correlation function $\xi(\omega)$ versus inhomogeneity parameter ω.

$$\xi(\omega)=\langle\{\mu\}|\vec{\sigma}_n\vec{\sigma}_{n+2}-1|\{\mu\}\rangle$$
$$=-\frac{3\omega^2+1}{\omega^2}I(\omega)-\frac{\omega^2-1}{\omega^2}I(0)-\frac{\omega^2+1}{\omega}\partial_\omega I(\omega), \tag{40}$$

where $I(\omega)$ is the integral which can be expressed via digamma functions, ψ, as

$$I(\omega)=\int_{-\infty}^{\infty}\frac{dx}{(x^2+1/4)\cosh\pi(x-\omega)}$$
$$=\psi\left(1+i\frac{\omega}{2}\right)-\psi\left(\frac{1}{2}+i\frac{\omega}{2}\right)+\psi\left(1-i\frac{\omega}{2}\right)-\psi\left(\frac{1}{2}-i\frac{\omega}{2}\right). \tag{41}$$

We see that though the translational invariance $n\to n+1$ is broken, the vacuum expectation value of the NNN exchange operator does not depend on the site parity. Up to the sign factor, the same is valid for the triple interaction terms: $\langle\vec{\sigma}_{2k}(\vec{\sigma}_{2k+1}\times\vec{\sigma}_{2k+2})\rangle=-\langle\vec{\sigma}_{2k-1}(\vec{\sigma}_{2k}\times\vec{\sigma}_{2k+1})\rangle$.

In the limit $\omega\to 0$, from Eq. (40), one will recover the known result for the expectation value of the NNN exchange operator of the Heisenberg *XXX* isotropic chain,

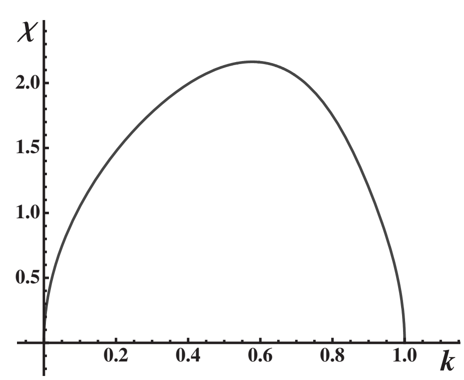

FIG. 2. Chiral-spin order parameter, χ, versus $k=\frac{\omega^2}{1+\omega^2}$.

$$\langle\vec{\sigma}_n\vec{\sigma}_{n+2}\rangle = 1 - 16\log(2) + 9\zeta(3). \quad (42)$$

This was calculated from the ground state energy of the Hubbard model in Ref. 14. We present the function $\xi(\omega)$ in Fig. 1.

The chiral-spin order parameter $\langle\{\mu\}|\vec{\sigma}_n(\vec{\sigma}_{n+1}\times\vec{\sigma}_{n+2})|\{\mu\}\rangle$ is also important, as it defines the measure of chirality of the state. Substituting the densities ρ and F for the ground state, Eqs. (38) and (39), into Eq. (35), we obtain

$$\chi(\omega) = (-1)^n\langle\vec{\sigma}_n(\vec{\sigma}_{n+1}\times\vec{\sigma}_{n+2})\rangle = \left[\frac{2}{\omega}I(0) - \frac{2+4\omega^2}{\omega}I(\omega) - (1+\omega^2)\partial_\omega I(\omega)\right]. \quad (43)$$

This function is plotted in Fig. 2. We see a perfect match between our plot and the numerical simulations of Ref. 12.

In conclusion, let us briefly comment on the possibility of extension of the developed method to the third neighbor correlation functions, in particular of the type, $\langle\vec{\sigma}_n\vec{\sigma}_{n+3}\rangle$. For this case one has to increase the level of inhomogeneity of the model by introducing two different shifts of the spectral parameter and consider the following monodromy matrix:

$$\Theta(\lambda;\omega_1,\omega_2) = L_{3N,a}(\lambda)L_{3N-1,a}(\lambda-\omega_1)L_{3N-2,a}(\lambda-\omega_2)\cdots L_{3,a}(\lambda)L_{2,a}(\lambda-\omega_1)L_{1,a}(\lambda-\omega_2), \quad (44)$$

which is defined on the lattice with $3N$ sites. With this construction, one again has an integrable model with commuting family of transfer matrices, but, contrary to case considered above, we will have now three local Hamiltonian operators. It is straightforward to derive their explicit forms, which are rather cumbersome and we do not bring them here. These operators contain different products of spins residing on four neighboring sites, including, e.g., the combination $\vec{\sigma}_n\vec{\sigma}_{n+3}$. The number of quasishift operators will be also three instead of the two as in Eq. (10). It is plausible to think that one can use relations analogous to Eqs. (18)–(20) in order to extract contributions of different summands out of the local Hamiltonian operators, at least in the homogeneous limit, $\omega_1=\omega_2=0$. These investigations may constitute a separate paper.

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[1] L. P. Kadanoff, Phys. Rev. Lett. **23**, 1430 (1969).
[2] K. Wilson, Phys. Rev. B **4**, 3174 (1971).
[3] K. Wilson, Phys. Rev. B **4**, 3184 (1971).
[4] L. P. Kadanoff, in *Cooperative Phenomena*, edited by H. Haken (North Holland, Amsterdam 1974), p. 139.
[5] V. V. Mkhitaryan and T. A. Sedrakyan (unpublished).
[6] F. D. M. Haldane, Phys. Rev. B **25**, 4925 (1982).
[7] X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989).
[8] L. D. Faddeev, arXiv:hep-th/9605187.
[9] C. Destri and H. J. de Vega, Nucl. Phys. B **290**, 363 (1987).
[10] A. A. Zvyagin, Phys. Rev. B **51**, 12579 (1995).
[11] V. Yu. Popkov and A. A. Zvyagin, Phys. Lett. A **175**, 295 (1993).
[12] H. Frahm and C. Roedenbeck, J. Phys. A **30**, 4467 (1997).
[13] D. Arnaudon, R. Poghossian, A. Sedrakyan, and P. Sorba, Nucl. Phys. B **558**, 638 (2000).
[14] M. Takahashi, J. Phys. C **10**, 1289 (1977).
[15] M. Jimbo and T. Miwa, *Algebraic Analysis of Solvable Lattice Models* (AMS, Providence, 1994).
[16] N. Kitanine, J. M. Maillet, and V. Terras, Nucl. Phys. B **554**, 647 (1999).
[17] N. Kitanine, J. M. Maillet, and V. Terras, Nucl. Phys. B **567**, 554 (2000).
[18] K. Sakai, M. Shiroishi, Y. Nishiyama, and M. Takahashi, Phys. Rev. E **67**, 065101(R) (2003).
[19] H. E. Boos, M. Shiroishi, and M. Takahashi, Nucl. Phys. B **712**, 573 (2005).
[20] G. Japaridze, A. Nersesyan, and P. Wiegmann, Nucl. Phys. B **230**, 511 (1984).
[21] H. Frahm and C. Roedenbeck, Eur. Phys. J. B **10**, 409 (1999).
[22] V. V. Mkhitaryan and A. G. Sedrakyan, Nucl. Phys. B **673**, 455 (2003).